Entropy production as a scale-dependent dynamical quantity

Entropy production $\Pi = dS/dt$ has been formalized as a trajectory-level dynamical variable with well-defined scaling behavior near non-equilibrium critical points, but it remains a derived composite quantity in every known framework — never appearing as a fundamental coupling constant, threshold parameter, or independent field in any action or Hamiltonian. The most important formal result is that entropy production rate (EPR) density diverges at criticality in active field theories below four dimensions with exponent **$\theta_\sigma = \nu(4-d)$**, where ν is the correlation-length exponent. Under iterative coarse-graining, EPR obeys the power-law **$\sigma(bL) = b^\alpha \sigma(L)$**, where the sign of α distinguishes genuinely non-equilibrium from equilibrium-like macroscopic behavior. DOI APS Journals However, equilibrium phase transitions — the entire Ising, XY, Heisenberg, and Potts families — exhibit sharp criticality with identically zero entropy production, establishing that Π is neither necessary nor sufficient for critical behavior. This report catalogs the explicit mathematical structures, scaling exponents, and RG treatments linking entropy production to dynamical criticality, alongside the strongest falsification arguments against entropy-scaling universality.

Stochastic thermodynamics establishes Π as a fluctuating trajectory-level observable

The modern framework of stochastic thermodynamics (Seifert 2012) defines entropy production along individual stochastic trajectories, not merely as an ensemble average. PubMed +2 For an overdamped Langevin system, the total entropy production rate decomposes as

$$\dot{S}_{\text{tot}} = \dot{S}_{\text{sys}} + \dot{S}_{\text{med}}, \quad \dot{S}_{\text{sys}} = -\frac{d}{dt}\int dx\, p \ln p, \quad \dot{S}_{\text{med}} = \langle \dot{q} \rangle / T$$

with the ensemble-level rate given by the manifestly non-negative expression **$\dot{S}_{tot} = \int dx\, j^2/(Dp) \geq 0$**, where j is the probability current and D the diffusion coefficient. Three major decomposition schemes refine this picture:

- **Esposito–Van den Broeck adiabatic/nonadiabatic split**: $\dot{S}_{tot} = \dot{S}_a + \dot{S}_{na}$, with both terms independently non-negative. Core The adiabatic part quantifies housekeeping dissipation maintaining the instantaneous NESS; the nonadiabatic part captures the thermodynamic cost of driving between steady states, connected to free-energy dissipation via $\dot{F} = -T\dot{S}_{na} + \dot{W}$.
- **Hatano–Sasa excess entropy**: For transitions between NESS, the excess entropy production $Y = \int d\tau\, (\partial\varphi/\partial\alpha)\cdot\dot{\alpha}$ satisfies the integral fluctuation theorem **$\langle e^{-Y} \rangle = 1$** and a generalized Clausius inequality $\langle Y \rangle \geq 0$. PubMed Central Recent work by Dechant et al. (2022) reveals a geometric structure: the slow-driving limit connects excess entropy to a **Fisher information metric** in parameter space.
- **Maes–Netočný entropy flux/frenesy decomposition**: The path-space action splits into a time-antisymmetric part S (entropy flux) and a time-symmetric part D (frenesy/dynamical activity). ScienceDirect Path probabilities take the form $P(\omega) = P_{ref}(\omega)\cdot\exp[-D(\omega) + S(\omega)/2]$. Crucially, **both** S and D are needed to characterize non-equilibrium dynamics — entropy production alone is insufficient. Kuleuven

The **fluctuation theorems** constrain the full distribution of Π: the Crooks relation P_F(ω)/P_R(−ω) = e^ω holds exactly for all driving strengths; threeplusone the Gallavotti–Cohen symmetry I(−σ) − I(σ) = σ governs the large-deviation rate function of time-averaged EPR. Springer Non-analyticity of I(σ) signals **dynamical phase transitions** in entropy production fluctuations, observed in driven lattice gases and exclusion processes.

A critical constraint on scale-dependence: **coarse-graining can only decrease entropy production** Cambridge Core arXiv (Seifert 2012; Esposito 2012). Coarse-grained EPR provides a lower bound on the true microscopic EPR, arXiv Cambridge Core and this monotonicity under coarse-graining holds for both the total and the adiabatic/nonadiabatic components.

Entropy production acquires critical scaling exponents in active field theories

The most significant formal results on EPR scaling at criticality come from active matter field theory:

Caballero and Cates (PRL 124, 240604, 2020) showed that in Active Model A (a non-conserved scalar φ^4 theory with time-reversal symmetry-breaking terms), even when the active couplings are **irrelevant in the RG sense** and the static universality class remains equilibrium Ising, the EPR density exhibits nontrivial critical scaling. This "stealth entropy production" has the EPR per spacetime correlation volume **Σ_V = σ · ξ^d · ξ^z** diverging on approach to criticality below d = 4, despite the critical exponents themselves being equilibrium values. DOI

Paoluzzi (PRE 105, 044139, 2022) derived the explicit critical exponent:

$$\sigma \sim |r|^{-\theta_\sigma}, \quad \theta_\sigma = \nu(4-d)$$

where r = (T−T_c)/T_c is the reduced temperature. In **d = 3**: θ_σ ≈ 0.63 (using Ising ν ≈ 0.63); in **d = 2**: θ_σ ≈ 2.0. Above d_c = 4, θ_σ < 0 and EPR vanishes at criticality (equilibrium-like). This result is formally derived from the RG, not analogical, but it applies specifically to active field theories where detailed balance is broken by explicit non-equilibrium couplings. arXiv

Falasco and Esposito (PRE 105, L042601, 2022) established the scaling of EPR under iterative coarse-graining for active disordered media: APS Journals

$$\sigma(bL) = b^\alpha\, \sigma(L)$$

The **sign of α** classifies macroscopic behavior: α > 0 means EPR grows with scale (genuinely non-equilibrium); α < 0 means EPR vanishes at large scales (equilibrium-like). DOI APS Journals This is the closest result to the hypothetical scaling law Π(λL) = λ^α Π(L).

Deffner and Zurek (PRE 96, 052125, 2017) derived Kibble–Zurek scaling of irreversible entropy production when driving through a phase transition at rate c: ResearchGate arXiv

$$\langle \Sigma_{\mathrm{irrev}} \rangle \sim c^{(2+\Lambda)/(1+z\nu)}$$

For the Ising universality class: ⟨Σ⟩ ~ c^{2/(1+zν)}. APS +2 Numerical verification for the KPZ universality class (ν = 1, z = 3/2) yielded exponent **0.78** versus the predicted **0.8**, confirming the scaling form. APS arXiv

RG treats dissipation coefficients but not entropy production as running couplings

In every known renormalization group treatment of non-equilibrium field theories, **dissipation coefficients renormalize but entropy production itself does not appear as a coupling constant with its own beta function.** This is a crucial distinction.

In the **Canet–Delamotte non-perturbative RG** for the KPZ equation $\partial_t h = \nu\nabla^2 h + (\lambda/2)(\nabla h)^2 + \eta$, the running couplings are the viscosity ν_k, nonlinear coupling λ_k, and noise strength D_k. The effective dimensionless coupling $g_k = K_d \lambda_k^2 D_k/\nu_k^3$ flows to a fixed point. The **viscosity ν acquires an anomalous dimension** — it is a scale-dependent dissipation coefficient. At the KPZ fixed point ($d = 1$): $z = 3/2$, $\chi = 1/2$. At the Edwards–Wilkinson fixed point: $z = 2$. A recently discovered "inviscid Burgers" fixed point at $g \to \infty$ with $z = 1$ corresponds to zero dissipation — the limit of vanishing viscosity is itself a distinct universality class. Psl

For the **Hohenberg–Halperin dynamic universality classes** (Models A through J), the kinetic coefficient Γ (Onsager coefficient) governs dissipation and renormalizes under RG. In Model A: $z = 2 + c\eta$; in Model B: $z = 4 - \eta$. However, these are equilibrium models satisfying detailed balance by construction, so **steady-state EPR is identically zero** and has no independent scaling dimension.

In the **Reggeon field theory** for directed percolation, the action $S[\tilde{\varphi},\varphi] = \int d^d x\, dt\, [\tilde{\varphi}(\partial t\, \varphi - D\nabla^2\varphi - \tau_0\varphi) + u_0(\tilde{\varphi}\varphi^2 - \tilde{\varphi}^2\varphi)]$ *violates detailed balance through the asymmetric coupling of* φ *and* $\tilde{\varphi}$. ScienceDirect *The diffusion constant D renormalizes under RG. Critical exponents at one-loop in* $\varepsilon = 4-d$: $\beta = 1 - \varepsilon/6$, $\nu_\perp = 1/2 + \varepsilon/16$, $z = 2 - \varepsilon/12$. **No published work explicitly computes the EPR scaling exponent at the DP critical point**, though one expects $\Pi \sim \rho_{stat} \sim |p-p_c|^\beta$ in the active phase. This is an important open problem.

System	Dynamic exponent z	EPR scaling	Status
Active Model A ($d < 4$)	$2 + c\eta$ (Ising)	$\sigma \sim \lvert r\rvert^{-\nu(4-d)}$ diverges	**Formal result**
Active Model A ($d > 4$)	2 (Gaussian)	$\sigma \to 0$ at criticality	**Formal result**
KZ driving through transition	z (system-dependent)	$\langle\Sigma\rangle \sim c^{2/(1+z\nu)}$	**Formal + verified**
Coarse-grained active matter	—	$\sigma(bL) = b^\alpha \sigma(L)$	**Formal result**
KPZ	3/2 (d=1)	Natural dim: $-(d+z)$; anomalous dim unknown	**Open problem**
Directed percolation	1.58 (d=1)	Expected $\sim \lvert p-p_c\rvert^\beta$; not computed	**Open problem**
Equilibrium HH Models A–J	varies	$\sigma = 0$ exactly (FDT satisfied)	**Exact**

Field-theoretic formulations treat entropy production as a derived functional

Four distinct field-theoretic frameworks incorporate entropy production, each with different mathematical status:

The MSR–Janssen–De Dominicis formalism maps stochastic dynamics onto path integrals with action $S[x,\tilde{x}] = \int i\tilde{x}[\partial_t x - F(x)] + (\sigma/2)H(x)^2\tilde{x}^2$. Entropy production is identified as the **antisymmetric part of the action under time reversal**: the log-ratio of forward and backward path measures. Springer When detailed balance holds, the MSR action possesses a BRST-type supersymmetry whose breaking signals nonzero EPR. arXiv In active field theories, the FDT appears as a Ward–Takahashi identity, and its violation quantifies entropy production arXiv spectrally: $\dot{S} = \int d^d q\, \sigma(q)$ (Nardini et al., Phys. Rev. X 7, 2017). arXiv

The GENERIC formalism (Grmela–Öttinger 1997) provides the most structured treatment via the equation $dx/dt = L(x)\cdot\delta E/\delta x + M(x)\cdot\delta S/\delta x$, where L is a Poisson operator (reversible) and M is a friction operator (irreversible, positive semidefinite). Entropy production takes the explicitly non-negative quadratic form $\boldsymbol{\Pi = (\delta S/\delta x)\cdot M\cdot(\delta S/\delta x) \geq 0}$, guaranteed by positive semidefiniteness of M. Wikipedia The framework is **postulated to hold at multiple scales** — from microscopic Hamiltonian dynamics to macroscopic hydrodynamics — with coarse-graining procedures derived from projection operators (Öttinger 1998). The multi-scale extension (Pavelka, Klika, Grmela 2018) treats GENERIC hierarchically, but entropy production remains a derived functional at each level, not an independent control parameter.

Wasserstein gradient flows (Jordan–Kinderlehrer–Otto 1998) cast the Fokker–Planck equation as the gradient flow of the entropy functional $E(\rho) = \int \rho \ln \rho\, dx$ in Wasserstein-2 space. The entropy production rate equals the **Fisher information**: $dE/dt = -I(\rho) = -\int \rho|\nabla \ln \rho|^2\, dx$. Otto calculus provides a Riemannian structure where the Wasserstein metric supplies "kinetic" structure and entropy supplies "potential" structure. This is rigorous mathematics (Ambrosio, Gigli, Savaré) with exact, not analogical, content.

The Onsager–Machlup functional provides the most direct Lagrangian embedding: for an Ornstein–Uhlenbeck process, $L_{OM} = (\dot{x} + \alpha x)^2/(4D)$, where the cross-term integrates to entropy change and the quadratic terms give dissipation. The Rayleigh dissipation function Φ relates directly: $\boldsymbol{T\sigma = 2\Phi = \Sigma_{ij}\, \zeta_{ij}\, \dot{q}_i\, \dot{q}_j}$. Onsager's variational principle — minimizing the Rayleighian $R = \Phi + \dot{F}$ — yields overdamped dynamics and automatically satisfies reciprocal relations.

A notable result in non-equilibrium Landau theory: **Aron and Chamon (SciPost Phys. 8, 2020)** showed that breaking detailed balance generates non-analytic terms in the NESS potential $V_{NESS}(\varphi) = a_2\varphi^2 + a_4\varphi^4 + c_\alpha|\varphi|^{2+\alpha}$, where α is determined by the environment's density of states. This produces bath-dependent critical exponents $\beta_{NESS} = 1/|\alpha-1|$ at mean field, departing from equilibrium universality classes. The non-analytic term vanishes at equilibrium ($c_\alpha \to 0$), directly connecting detailed-balance violation to modified phase transition behavior.

Cross-domain evidence reveals structural parallels but not mathematical identity

Entropy production exhibits structural parallels across physical scales, but the mathematical definitions are domain-specific and not formally equivalent.

Quantum systems: Spohn's theorem (1978) establishes $\sigma\_EP(\rho) = -Tr(L(\rho)(\ln \rho - \ln \sigma)) \geq 0$ for any Lindblad semigroup with stationary state σ, Tum with $\sigma\_EP = 0$ iff $\rho = \sigma$. This is a formal result deriving from contractivity of quantum relative entropy. The quantum fluctuation theorem $\langle e^{-\sigma} \rangle = 1$ holds for arbitrary Lindblad dynamics (De Chiara and Imparato 2021). Near quantum critical points, EPR follows Kibble–Zurek scaling: $\sigma \propto \tau\_Q^{-d\nu/(1+z\nu)}$. Goes et al. (2020) showed that **dynamical criticality accelerates entropy production** in the Lipkin–Meshkov–Glick model. American Physical Society +2

Gravitational systems: The Bekenstein–Hawking entropy $S\_BH = A/(4l\_P^2)$ Scholarpedia and Hawking's area theorem Springer $dA/dt \geq 0$ provide a formally derived gravitational second law. The generalized second law PubMed Central $d(S\_BH + S\_matter)/dt \geq 0$ extends this to quantum processes. Penrose's Weyl curvature hypothesis — that gravitational entropy is encoded in C_abcd with the scalar $P^2 = C\_abcd\ C^{abcd}/(R\_ab\ R^{ab})$ increasing monotonically — remains a **conjecture** without rigorous derivation. arXiv

Neural systems: Beggs and Plenz (2003) observed neuronal avalanches with power-law distributions $P(s) \propto s^{-3/2}$ and $P(T) \propto T^{-2}$, consistent with critical branching processes. At criticality (branching ratio $\sigma = 1$), information entropy is maximized across scales and dynamic range is optimized (Kinouchi and Copelli 2006). However, the connection to **thermodynamic** entropy production remains largely analogical — neural criticality studies primarily use information-theoretic entropy measures, not dissipation rates.

Biology: The maximum entropy production principle has been applied to climate (Paltridge 1975), ecology, and metabolism, but Vallino (2010) showed biological systems maximize **long-term spatiotemporally averaged** entropy production, not instantaneous rates. For bistable chemical systems (Schlögl model), Nicoli et al. (2017) proved the state with highest entropy production is most probable: Nature $\ln(P\_i/P\_j) \propto (\Pi\_i - \Pi\_j)$.

The deepest mathematical unification across domains lies in **relative entropy** $D(\rho\|\sigma)$: quantum ($Tr\ \rho(\ln \rho - \ln \sigma)$), classical (KL divergence), and gravitational (generalized entropy). In each domain, entropy production is the negative time derivative of a relative entropy functional, and non-negativity follows from contractivity under physical evolution. arXiv This is formal mathematics, not analogy. Whether this mathematical parallel reflects a deeper physical principle remains open.

Strong falsification arguments constrain universality claims

The case against entropy production as a universal structural driver of criticality is formidable on multiple fronts.

Equilibrium criticality requires zero entropy production. The Ising model' Wikipedia s exact solution Wikipedia (Onsager 1944), the entire Wilson–Fisher RG framework, and all equilibrium universality classes derive sharp phase transitions, divergent correlation lengths, and universal critical exponents with $\Pi \equiv 0$. Criticality emerges from competition between energy and entropy in the free energy $F = E - TS$, not from dissipation. This is the single strongest objection: entropy production is neither necessary nor sufficient for critical phenomena.

Entropy production can peak away from critical points. Aguilera, Igarashi, and Shimazaki (Nature Communications 2023) showed in the asymmetric Sherrington–Kirkpatrick model that EPR is "significantly larger for quasi-deterministic disordered dynamics" in the low-temperature regime — **Π is maximized far from the critical point**. Multiple thermodynamic quantities are needed to disambiguate criticality from merely disordered high-dissipation states.

Coarse-graining dependence undermines universality. Entropy production is fundamentally observer-dependent: MDPI "what appears irreversible at one scale may appear nearly reversible at another" arXiv (arXiv:2602.15663, 2026). Finer temporal resolution systematically decreases measured EPR; finer spatial resolution increases it. arXiv ResearchGate For active systems with generalized Langevin dynamics, integrating out degrees of freedom can either increase or decrease EPR, violating even monotonicity. ResearchGate If Π depends on the description level, claims about its universal structural role require specifying which Π at which scale.

MEPP faces fundamental epistemological problems. Dyke and Kleidon (2010) argued MEPP is unfalsifiable: any disagreement can be attributed to incorrect application rather than failure of the principle. Dewar (2009) showed MEP is equivalent to Jaynes' MaxEnt inference — a statistical procedure with no causal explanatory power. Landauer (1975) demonstrated failure of entropy production principles for simple electrical circuits with temperature inhomogeneities.

Breaking detailed balance does not imply structure. A particle diffusing in a constant rotational force field permanently breaks detailed balance with positive EPR but exhibits no criticality, no phase transition, and no emergent structure. Simple three-state Markov chains can have circulating probability currents and nonzero Π while remaining entirely featureless.

Entropy production definitions differ fundamentally across domains. Stochastic thermodynamics defines Π via log-ratios of path probabilities APS Journals in Markov processes. In QFT, entropy requires choosing truncation schemes and is complicated by UV divergences. In quantum mechanics, von Neumann entropy is nonlinear in the density matrix and thus not a physical observable in a single-shot sense. In gravitational systems, entropy production beyond weak-field approximations remains undefined. These are not merely different representations of the same quantity — they involve genuinely distinct mathematical structures.

What has been formally established versus what remains analogical

The following table summarizes the status of each claimed connection:

Claim	Mathematical status	Key result
EPR has critical scaling exponent $\theta_\sigma = \nu(4-d)$	**Formal derivation** (active φ^4 theories)	Caballero-Cates 2020; Paoluzzi 2022
EPR obeys coarse-graining scaling $\sigma(bL) = b^\alpha \sigma(L)$	**Formal derivation** (active disordered media)	Falasco-Esposito 2022
Irreversible EP follows KZ scaling	**Formal + numerically verified**	Deffner-Zurek 2017
Dissipation coefficients renormalize under RG	**Formal** (all dynamical field theories)	Canet et al.; Hohenberg-Halperin
EP has its own RG beta function	**Not established** in any framework	Open question
EP modulates coupling constants or thresholds	**Not established** as a formal result	Analogy only
GENERIC gives multi-scale EP framework	**Formal** (postulated, not derived from first principles)	Grmela-Öttinger 1997
Non-eq Landau terms arise from DB breaking	**Formal derivation**	Aron-Chamon 2020
Neural criticality optimizes EP	**Analogical** (information-theoretic, not thermodynamic)	Beggs-Plenz 2003
MEPP as universal principle	**Debated/unfalsifiable**	Multiple criticisms
Relative entropy unifies EP across domains	**Formal mathematics** (structural parallel)	Spohn 1978; KL divergence
Gravitational entropy production (Weyl hypothesis)	**Conjecture**	Penrose 1979

Conclusion

The mathematical landscape connecting entropy production to scale-dependent dynamical criticality is rich but fragmented. **Three genuinely formal results stand out**: the critical exponent $\theta_\sigma = \nu(4-d)$ for EPR in active field theories, APS Journals the coarse-graining scaling law $\sigma(bL) = b^\alpha \sigma(L)$, APS Journals and the Kibble–Zurek scaling of irreversible entropy production. APS +2 These demonstrate that EPR, while always a composite derived quantity, possesses well-defined and computable scaling behavior near non-equilibrium critical points. The GENERIC formalism provides the most structured multi-scale treatment, Wikipedia American Physical Society but entropy production remains a functional of state variables rather than an independent dynamical field.

The critical gap in the literature is the absence of an RG framework where entropy production itself runs as a coupling constant. Dissipation coefficients (viscosity ν, kinetic coefficient Γ, diffusion D) renormalize in every dynamical field theory, but these are individual transport coefficients — not the thermodynamic entropy production rate, which is a composite observable built from correlators. No beta function $\beta(\Pi)$ has been derived in any framework. The distinction matters: a running coupling actively determines the form of interactions at each scale, while a composite observable merely inherits scaling from its constituent fields.

The strongest barriers to universality are not technical but structural. Equilibrium criticality exists without entropy production; Wikipedia EPR is observer-dependent through coarse-graining; DOI concrete counterexamples show EPR maximized away from critical points; and the mathematical definitions of "entropy production" across quantum, gravitational, and classical domains are genuinely inequivalent. The relative entropy $D(\rho\|\sigma)$ provides the deepest formal bridge, but structural similarity of the mathematics does not establish physical equivalence of the phenomena. A defensible position is that entropy production is a **diagnostic observable** that exhibits nontrivial and computable scaling at non-equilibrium critical points — but not a universal structural driver of criticality itself.